last change 21.01.13@9.00
Gauss Factorization: solving linear equations

Factorization of matrix into prod of lotri & uptri
by Gauss elimination

$ax + by = u$
$cx + dy = v$

$x + \frac{b}{a}y = \frac{u}{a}$
$x + \frac{d}{c}y = \frac{v}{c}$

$x + \frac{b}{a}y = \frac{u}{a}$
$0 + \left(\frac{d}{c} - \frac{b}{a}\right) y = \frac{v}{c} - \frac{u}{a}$

in matrix form: **Gauss Factorization**

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} u \\ v \end{bmatrix}$$

triangularization: $\nabla \mathbf{x} = \mathbf{\Delta u}$

$$\begin{bmatrix} 1 & \frac{b}{a} \\ 0 & \left(\frac{d}{c} - \frac{b}{a}\right) \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \frac{1}{a} & 0 \\ -\frac{1}{a} & \frac{1}{c} \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix}$$

inverse of uptri is uptri: $\nabla^{-1} = \nabla$

$$\begin{bmatrix} 1 & \frac{b}{a} \\ 0 & \left(\frac{d}{c} - \frac{b}{a}\right) \end{bmatrix}^{-1} = \begin{bmatrix} 1 & -b\frac{c}{ad-bc} \\ 0 & a\frac{c}{ad-bc} \end{bmatrix}$$

sol factored into tria-mats: $\mathbf{x} = \nabla \mathbf{\Delta}$

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 & -b\frac{c}{ad-bc} \\ 0 & a\frac{c}{ad-bc} \end{bmatrix} \begin{bmatrix} \frac{1}{a} & 0 \\ -\frac{1}{a} & \frac{1}{c} \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix}$$

check

$$\begin{bmatrix} 1 & -b\frac{c}{ad-bc} \\ 0 & a\frac{c}{ad-bc} \end{bmatrix} \begin{bmatrix} \frac{1}{a} & 0 \\ -\frac{1}{a} & \frac{1}{c} \end{bmatrix} = \begin{bmatrix} \frac{d}{ad-bc} & -\frac{b}{ad-bc} \\ -\frac{c}{ad-bc} & \frac{a}{ad-bc} \end{bmatrix} = B^{-1}$$

or

$$\begin{bmatrix} \frac{1}{a} & 0 \\ -\frac{1}{a} & \frac{1}{c} \end{bmatrix}^{-1} \begin{bmatrix} 1 & -b\frac{c}{ad-bc} \\ 0 & a\frac{c}{ad-bc} \end{bmatrix}^{-1} = \begin{bmatrix} a & 0 \\ c & c \end{bmatrix} \begin{bmatrix} 1 & \frac{1}{a}b \\ 0 & \frac{1}{ac}\left(ad - bc\right) \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} = B$$

alternatively, using matrix operations

$$B = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

uptriangularization:

$$\begin{bmatrix} a^{-1} & 0 \\ 0 & c^{-1} \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 1 & \frac{1}{a}b \\ 1 & \frac{1}{c}d \end{bmatrix}$$

$$\begin{bmatrix}1 & 0\\ -1 & 1\end{bmatrix}\begin{bmatrix}1 & \frac{1}{a}b\\ 1 & \frac{1}{c}d\end{bmatrix} = \begin{bmatrix}1 & \frac{1}{a}b\\ 0 & \frac{1}{c}d - \frac{1}{a}b\end{bmatrix} \text{ uptri}$$

$\Longrightarrow$

$$\begin{bmatrix}1 & 0\\ -1 & 1\end{bmatrix}\begin{bmatrix}a^{-1} & 0\\ 0 & c^{-1}\end{bmatrix} = \begin{bmatrix}\frac{1}{a} & 0\\ -\frac{1}{a} & \frac{1}{c}\end{bmatrix} \text{ lotri}$$

hence: $\Delta\ B = \nabla$

$$\begin{bmatrix}\frac{1}{a} & 0\\ -\frac{1}{a} & \frac{1}{c}\end{bmatrix}\begin{bmatrix}a & b\\ c & d\end{bmatrix} = \begin{bmatrix}1 & \frac{1}{a}b\\ 0 & \frac{1}{c}d - \frac{1}{a}b\end{bmatrix}$$

also $\boldsymbol{\Delta}^{-1} = \boldsymbol{\Delta}$ and $\nabla^{-1} = \nabla$

$$\begin{bmatrix}\frac{1}{a} & 0\\ -\frac{1}{a} & \frac{1}{c}\end{bmatrix}^{-1} = \begin{bmatrix}a & 0\\ c & c\end{bmatrix}$$

hence $B = \boldsymbol{\Delta}\nabla$

$$\begin{bmatrix}a & 0\\ c & c\end{bmatrix}\begin{bmatrix}1 & \frac{1}{a}b\\ 0 & \frac{1}{c}d - \frac{1}{a}b\end{bmatrix} = \begin{bmatrix}a & b\\ c & d\end{bmatrix}$$

Cholesky factorization: suppose symmetry: $c = b$
by symmetric row and column operations generate identity
$CBC' = I \implies B = C^{-1}C$

start

$$\begin{bmatrix}\frac{1}{\sqrt{a}} & 0\\ 0 & 1\end{bmatrix}\begin{bmatrix}a & b\\ b & d\end{bmatrix}\begin{bmatrix}\frac{1}{\sqrt{a}} & 0\\ 0 & 1\end{bmatrix} = \begin{bmatrix}1 & \frac{b}{\sqrt{a}}\\ \frac{b}{\sqrt{a}} & d\end{bmatrix}$$

generate off diag zeros:

$$\begin{bmatrix}\frac{b}{\sqrt{a}} & 0\\ 0 & 1\end{bmatrix}\begin{bmatrix}1 & \frac{b}{\sqrt{a}}\\ \frac{b}{\sqrt{a}} & d\end{bmatrix}\begin{bmatrix}\frac{b}{\sqrt{a}} & 0\\ 0 & 1\end{bmatrix} = \begin{bmatrix}\frac{1}{a}b^2 & \frac{1}{a}b^2\\ \frac{1}{a}b^2 & d\end{bmatrix}$$

$$\begin{bmatrix}\frac{b}{\sqrt{a}} & 0\\ 0 & 1\end{bmatrix}\begin{bmatrix}\frac{1}{\sqrt{a}} & 0\\ 0 & 1\end{bmatrix}\begin{bmatrix}a & b\\ b & d\end{bmatrix}\begin{bmatrix}\frac{1}{\sqrt{a}} & 0\\ 0 & 1\end{bmatrix}\begin{bmatrix}\frac{b}{\sqrt{a}} & 0\\ 0 & 1\end{bmatrix} = \begin{bmatrix}\frac{1}{a}b^2 & \frac{1}{a}b^2\\ \frac{1}{a}b^2 & d\end{bmatrix}$$

$$\begin{bmatrix}1 & 0\\ -1 & 1\end{bmatrix}\begin{bmatrix}\frac{b}{\sqrt{a}} & 0\\ 0 & 1\end{bmatrix}\begin{bmatrix}\frac{1}{\sqrt{a}} & 0\\ 0 & 1\end{bmatrix}\begin{bmatrix}a & b\\ b & d\end{bmatrix}\begin{bmatrix}\frac{1}{\sqrt{a}} & 0\\ 0 & 1\end{bmatrix}\begin{bmatrix}\frac{b}{\sqrt{a}} & 0\\ 0 & 1\end{bmatrix}\begin{bmatrix}1 & -1\\ 0 & 1\end{bmatrix} = \begin{bmatrix}\frac{1}{a}b^2 & 0\\ 0 & d - \frac{1}{a}b^2\end{bmatrix}$$

$\Longrightarrow$

$$\underbrace{\begin{bmatrix}\frac{1}{\sqrt{\frac{1}{a}b^2}} & 0\\ 0 & \frac{1}{\sqrt{d-\frac{1}{a}b^2}}\end{bmatrix}\begin{bmatrix}1 & 0\\ -1 & 1\end{bmatrix}\begin{bmatrix}\frac{b}{\sqrt{a}} & 0\\ 0 & 1\end{bmatrix}\begin{bmatrix}\frac{1}{\sqrt{a}} & 0\\ 0 & 1\end{bmatrix}}_{C}$$

$$\begin{bmatrix}a & b\\ b & d\end{bmatrix}$$

$$\underbrace{\begin{bmatrix}\frac{1}{\sqrt{a}} & 0\\ 0 & 1\end{bmatrix}\begin{bmatrix}\frac{b}{\sqrt{a}} & 0\\ 0 & 1\end{bmatrix}\begin{bmatrix}1 & -1\\ 0 & 1\end{bmatrix}\begin{bmatrix}\frac{1}{\sqrt{\frac{1}{a}b^2}} & 0\\ 0 & \frac{1}{\sqrt{d-\frac{1}{a}b^2}}\end{bmatrix}}_{C'} = \begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}$$

$$\begin{bmatrix} \frac{1}{\sqrt{\frac{1}{a}b^2}} & 0 \\ 0 & \frac{1}{\sqrt{d-\frac{1}{a}b^2}} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} \frac{b}{\sqrt{a}} & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{a}} & 0 \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1}{a}\frac{b}{\sqrt{\frac{1}{a}b^2}} & 0 \\ -\frac{1}{a}\frac{b}{\sqrt{d-\frac{1}{a}b^2}} & \frac{1}{\sqrt{d-\frac{1}{a}b^2}} \end{bmatrix} = C$$

$$\begin{bmatrix} \frac{1}{\sqrt{a}} & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \frac{b}{\sqrt{a}} & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{\frac{1}{a}b^2}} & 0 \\ 0 & \frac{1}{\sqrt{d-\frac{1}{a}b^2}} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1}{a}\frac{b}{\sqrt{\frac{1}{a}b^2}} & -\frac{1}{a}\frac{b}{\sqrt{d-\frac{1}{a}b^2}} \\ 0 & \frac{1}{\sqrt{d-\frac{1}{a}b^2}} \end{bmatrix} = C'$$

$$\Longrightarrow$$

$$C \begin{bmatrix} a & b \\ b & d \end{bmatrix} C' = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \Longrightarrow \begin{bmatrix} a & b \\ b & d \end{bmatrix} = C^{-1} \left(C'\right)^{-1}$$

Application: Simulate normal random vector with a given covariance structure

next

Spectral decomp
motivation: princ axis transf
diagonalization of quadratic forms

$Ae = \lambda e$