last change: 25.12.2012@12.00

Exercises

---

1. log-linear regression models

$$\mathbb{P}\left(y_i \mid \alpha, \beta, \mathbf{x}_i\right) \sim \mu_i^{y_i} \exp\left(-\mu_i\right)$$
$$\mu\left(\mathbf{x}\right) = \exp\left(\alpha + \sum_j \beta^{(j)} x^{(j)}\right)$$
$$\log\left(\mu\left(\mathbf{x}\right)\right) = \alpha + \sum_j \beta^{(j)} x^{(j)}$$

data

1	y_1	$x_1^{(1)}$	...	$x_1^{(k)}$
⋮				
n	y_n	$x_n^{(1)}$		$x_n^{(k)}$

Likelihood:

$$\log L\left(\alpha, \beta \mid y_i, \mathbf{x}_i\right) = \sum_{i=1}^{n} y_i \left(\alpha + \sum_{j=1}^{k} \beta^{(j)} x_i^{(j)}\right) - \exp\left(\alpha + \sum_{j=1}^{k} \beta^{(j)} x_i^{(j)}\right)$$

log-linear models and contingency tables

for the following data:

age-group	cat1$\left(y^{(1)}\right)$	cat2$\left(y^{(2)}\right)$	cat3$\left(y^{(3)}\right)$
1	$y_1^{(1)}$	$y_1^{(2)}$	$y_1^{(3)}$
2	$y_2^{(1)}$	$y_2^{(2)}$	$y_2^{(3)}$
⋮			
I	$y_I^{(1)}$	$y_I^{(2)}$	$y_I^{(3)}$

a) specify Poisson reg model and interpret the coefficients

b) write likelihood function

c) write likelihood-ratio test for hypothesis of age–group independence of cat

d) discuss equivalence of the log-linear and multinomial models for the given grouped data

---

2. three-dimensional contingency tables and log-linear models

a) specify a 3-dimensional contingency table in (group) variables (factors) R, C, L and write an equivalent (full) log-linear model

b) interpret triple interaction as differences between pairwise interactions of any two variables at different levels of the third variable

c) interpret the pairwise interactions in terms of (conditional) odds

d) describe the LR-test for the hypothesis that triple interactions are negligible.

in the homogenuos odds ratio model:

e) interpret the pairwise interactions in terms of odds.

f) discuss the relation between pairwise interactions and conditional independence properties.

g) discuss the relations between the reduced model and a multinomial regression of first on second and third factors (hint: Table 7.2)

---

3. find an example to show

$$\mathbb{P}(x \mid y, z) = \mathbb{P}(x \mid z) \nRightarrow \mathbb{P}(z \mid y, x) = \mathbb{P}(z \mid x)$$

---

4. define

$$OR(X \mid Y) = \frac{\mathbb{P}(X = x_{i_1} \mid y_{j_1}) / \mathbb{P}(X = x_{i_2} \mid y_{j_1})}{\mathbb{P}(X = x_{i_1} \mid y_{j_1}) / \mathbb{P}(X = x_{i_2} \mid y_{j_2})}$$

show

$$OR(X \mid Y) = OR(Y \mid X)$$

---

5. define

$$OR(X \mid Y, Z) = \frac{\mathbb{P}(X = x_{i_1} \mid y_{j_1}, z_{k_1}) / \mathbb{P}(X = x_{i_2} \mid y_{j_1}, z_{k_1})}{\mathbb{P}(X = x_{i_1} \mid y_{j_1}, z_{k_1}) / \mathbb{P}(X = x_{i_2} \mid y_{j_2}, z_{k_2})} = \frac{p_{i_1 \mid j_1 k_1} \big/ p_{i_2 \mid j_1 k_1}}{p_{i_1 \mid j_2 k_2} \big/ p_{i_2 \mid j_2 k_2}}$$

$$= \frac{p_{i_1 j_1 k_1} \,/\, p_{i_2 j_1 k_1}}{p_{i_1 j_2 k_2} \,/\, p_{i_2 j_2 k_2}}$$

note that in loglinear models with only pairwise interactions:

$$\begin{aligned}
\log OR(X \mid Y, Z) &= \Big(\alpha_{i_1} + \beta_{j_1} + \gamma_{k_1} + (\alpha\beta)_{i_1 j_1} + (\alpha\gamma)_{i_1 k_1} + (\beta\gamma)_{j_1 k_1}\Big) \\
&\quad - \Big(\alpha_{i_2} + \beta_{j_1} + \gamma_{k_1} + (\alpha\beta)_{i_2 j_1} + (\alpha\gamma)_{i_2 k_1} + (\beta\gamma)_{j_1 k_1}\Big) \\
&\quad - \Big(\alpha_{i_1} + \beta_{j_2} + \gamma_{k_2} + (\alpha\beta)_{i_1 j_2} + (\alpha\gamma)_{i_1 k_2} + (\beta\gamma)_{j_2 k_2}\Big) \\
&\quad + \Big(\alpha_{i_2} + \beta_{j_2} + \gamma_{k_2} + (\alpha\beta)_{i_2 j_2} + (\alpha\gamma)_{i_2 k_2} + (\beta\gamma)_{j_2 k_2}\Big) \\
&= \Big((\alpha\beta)_{i_1 j_1} + (\alpha\gamma)_{i_1 k_1} + (\beta\gamma)_{j_1 k_1}\Big) - \Big((\alpha\beta)_{i_2 j_1} + (\alpha\gamma)_{i_2 k_1} + (\beta\gamma)_{j_1 k_1}\Big) \\
&\quad - \Big((\alpha\beta)_{i_1 j_2} + (\alpha\gamma)_{i_1 k_2} + (\beta\gamma)_{j_2 k_2}\Big) + \Big((\alpha\beta)_{i_2 j_2} + (\alpha\gamma)_{i_2 k_2} + (\beta\gamma)_{j_2 k_2}\Big) \\
&= \Big((\alpha\beta)_{i_1 j_1} + (\alpha\gamma)_{i_1 k_1}\Big) - \Big((\alpha\beta)_{i_2 j_1} + (\alpha\gamma)_{i_2 k_1}\Big) \\
&\quad - \Big((\alpha\beta)_{i_1 j_2} + (\alpha\gamma)_{i_1 k_2}\Big) + \Big((\alpha\beta)_{i_2 j_2} + (\alpha\gamma)_{i_2 k_2}\Big)
\end{aligned}$$

---

6. if $(X \perp Y) \mid Z \implies OR(X \mid Y, Z) = OR(X \mid Z)$

a) by symmetry find the corresponding OR property for $(Z \perp Y) \mid X$
b) show that if $(X \perp Y) \mid Z$ or $(Z \perp Y) \mid X$ the X, Y, Z table is collapsible in Y.

---

3. corresponding to the data

```
data
      A   B   C   D
cat1 42  44  40 126
cat2 33 133  43  37
cat3 37  43 127  46
cat4 48  42  38 121
```

interpret the following outputs:

a) model

```
Coefficients:
            Estimate Std. Error z value Pr(>|z|)    
(Intercept)  3.68888    0.07906  46.661  < 2e-16 ***
xB           0.49317    0.10033   4.915 8.86e-07 ***
xC           0.43825    0.10140   4.322 1.55e-05 ***
xD           0.72392    0.09633   7.515 5.71e-14 ***\bigskip
```

b) model

```
Coefficients:
            Estimate Std. Error z value Pr(>|z|)    
(Intercept)  3.69685    0.09601  38.504  < 2e-16 ***
xB           0.49317    0.10033   4.915 8.86e-07 ***
xC           0.43826    0.10140   4.322 1.55e-05 ***
xD           0.72392    0.09633   7.515 5.71e-14 ***
ycat2       -0.02410    0.08963  -0.269    0.788    
ycat3        0.00396    0.08900   0.044    0.965    
ycat4       -0.01198    0.08936  -0.134    0.893    \bigskip
```

c) model

```
Coefficients:
            Estimate Std. Error z value Pr(>|z|)    
(Intercept)  3.73767    0.15430  24.223  < 2e-16 ***
xB           0.04652    0.21572   0.216 0.829264    
xC          -0.04879    0.22093  -0.221 0.825216    
xD           1.09861    0.17817   6.166 7.01e-10 ***
ycat2       -0.24116    0.23262  -1.037 0.299868    
ycat3       -0.12675    0.22547  -0.562 0.574002    
ycat4        0.13353    0.21129   0.632 0.527396    
xB:ycat2     1.34732    0.29045   4.639 3.50e-06 ***
xC:ycat2     0.31348    0.31995   0.980 0.327192    
xD:ycat2    -0.98420    0.29846  -3.298 0.000975 ***
xB:ycat3     0.10376    0.31116   0.333 0.738779    
xC:ycat3     1.28206    0.28933   4.431 9.37e-06 ***
xD:ycat3    -0.88089    0.28375  -3.104 0.001906 ** 
xB:ycat4    -0.18005    0.30196  -0.596 0.550991    
xC:ycat4    -0.18482    0.30977  -0.597 0.550743    
xD:ycat4    -0.17402    0.24667  -0.706 0.480498    
```