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Exercises

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1. binomial logit

for the following data (X group variable)

	Y		
		0	1
X	x_1	n_{11}	n_{12}
	$\vdots$		
	x_J	n_{J1}	n_{J2}

a) write a binary regression model

solution:
general:

$$\log\left(\mathbb{P}\left(y=j\mid x\right)/\,\mathbb{P}\left(y=1\mid x\right)\right)=\alpha_j+\sum\nolimits_{i=1}^{I}\beta_{ij}I\left(x\in\text{group}\left(i\right)\right)$$
$$\alpha_1=\beta_{1j}=\beta_{i1}=0$$

binomial ($j=0,1$):

$$\text{logit}\left(\mathbb{P}\left(y=1\mid x\right)\right)=\alpha+\sum\nolimits_{i>1}^{I}\beta_iI\left(x\in\text{group}\left(i\right)\right)$$

or

$$\mathbb{P}\left(y\mid x\right)=\begin{cases}\frac{\exp\alpha+\sum_{i>1}^{I}\beta_iI(x\in\text{group}(i))}{1+\exp\alpha+\sum_{i>1}^{I}\beta_iI(x\in\text{group}(i))} & \text{for } y=1\\ \frac{1}{1+\exp\alpha+\sum_{i>1}^{I}\beta_iI(x\in\text{group}(i))} & \text{for } y=0\end{cases}$$

b) interpret the regression coefficients for the logit model

solution:
general

$$\beta_{ij}=\underbrace{\text{logit}\left(\mathbb{P}\left(y=j\mid\text{group}\left(i\right)\right)\right)}_{\alpha_j+\beta_{ij}}-\underbrace{\text{logit}\left(\mathbb{P}\left(y=1\mid\text{group}\left(1\right)\right)\right)}_{\alpha_j}$$
$$=\log\left(\frac{\mathbb{P}\left(y=j\mid\text{group}\left(i\right)\right)\;/\,\mathbb{P}\left(y=1\mid\text{group}\left(i\right)\right)}{\mathbb{P}\left(y=j\mid\text{group}\left(1\right)\right)\;/\,\mathbb{P}\left(y=1\mid\text{group}\left(1\right)\right)}\right)$$

binomial ($j=0,1$):

$$\beta_i = \underbrace{\text{logit}\left(\mathbb{P}\left(y=1 \mid \text{group}\left(i\right)\right)\right)}_{\alpha+\beta_i} - \underbrace{\text{logit}\left(\mathbb{P}\left(y=1 \mid \text{group}\left(1\right)\right)\right)}_{\alpha}$$
$$= \log\left(\frac{\mathbb{P}\left(y=1 \mid \text{group}\left(i\right)\right) \ / \, \mathbb{P}\left(y=0 \mid \text{group}\left(i\right)\right)}{\mathbb{P}\left(y=1 \mid \text{group}\left(1\right)\right) \ / \, \mathbb{P}\left(y=0 \mid \text{group}\left(1\right)\right)}\right)$$

in short form

$$\beta_i = \log\frac{p_{1|i} / p_{0\ |i}}{p_{1|1} / p_{0|1}}$$

c) write an equivalent log-linear model

solution:

$$\log\mu_{ij} = \mu_0 + \alpha_i + \beta_j + (\alpha\beta)_{ij}$$
$$\alpha_1 = \beta_1 = (\alpha\beta)_{1j} = (\alpha\beta)_{i1} = 0$$

equivalence

$$\mu_{ij} = \mu p_{ij}$$
$$\mu = \sum\nolimits_{ij} \mu_{ij}$$

$\Longrightarrow$

$$\text{logit} p_{2\ /i} = \log\frac{p_{2/i}}{p_{1/i}} = \log\frac{p_{i2}}{p_{12}} = \log\frac{\mu_{2i}}{\mu_{10}} = \alpha_2 + \beta_j + (\alpha\beta)_{2j} - \beta_j = \alpha_2 + (\alpha\beta)_{2j}$$

$\Longrightarrow$

$$\log\frac{p_{2j}}{p_{1j}} - \log\frac{p_{21}}{p_{11}} = (\alpha\beta)_{2j}$$

$\Longrightarrow$

$$(\alpha\beta)_{2j} = \log\left(\frac{\mathbb{P}\left(y=1 \mid \text{group}\left(j\right)\right) \ / \, \mathbb{P}\left(y=0 \mid \text{group}\left(1\right)\right)}{\mathbb{P}\left(y=1 \mid \text{group}\left(j\right)\right) \ / \, \mathbb{P}\left(y=0 \mid \text{group}\left(1\right)\right)}\right)$$

d) compare the coefficients for the log-linear and for the logit model

solution:

$$\log\mu_{ij} = \mu_0 + \alpha_i + \beta_j + (\alpha\beta)_{ij}$$
$$\alpha_1 = \beta_1 = (\alpha\beta)_{1j} = (\alpha\beta)_{i1} = 0$$

hence (see manuscript)

$$\log\left(p_{j|i}\ /\ p_{1|i}\right) = \alpha_j^* + \beta_{ij}^*$$
$$\alpha_j^* = \beta_j$$
$$\beta_{ij}^* = (\alpha\beta)_{ij}$$

special case binomial($j = 0, 1$)

$$\log\left(p_{1|i}\ /\ p_{0|i}\right) = \text{logit}p_{1|i} = \alpha^* + \beta_i^*$$

with $\text{logit}(p_{j|i}) - \text{logit}(p_{j|1}) = \beta_i^*$

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2. continuous regressors multinomial logit

For the following data

case	age$(x^{(1)})$	inc$(x^{(2)})$	cat1$(y^{(1)})$	cat2$(y^{(2)})$	cat3$(y^{(3)})$
1	$x_1^{(1)}$	$x_1^{(2)}$	$y_1^{(1)}$	$y_1^{(2)}$	$y_1^{(3)}$
2	$x_2^{(1)}$	$x_2^{(2)}$	$y_2^{(1)}$	$y_2^{(2)}$	$y_2^{(3)}$
$\vdots$					
I	$x_I^{(1)}$	$x_I^{(2)}$	$y_I^{(1)}$	$y_I^{(2)}$	$y_I^{(3)}$

a) specify multinomial reg and interpret the regression coefficients

solution:

$$\mathbb{P}\left(y = j \mid x^{(1)}, x^{(2)}\right) = \begin{cases} \frac{\exp\left(\alpha_j + \beta_j^{(1)} x^{(1)} + \beta_j^{(2)} x^{(2)}\right)}{1 + \sum_{k>1} \exp\left(\alpha_k + \beta_k^{(1)} x^{(1)} + \beta_k^{(2)} x^{(2)}\right)} & \text{if } j > 1 \\ \frac{1}{1 + \sum_{k>1} \exp\left(\alpha_k + \beta_k^{(1)} x^{(1)} + \beta_k^{(2)} x^{(2)}\right)} & \text{if } j = 1 \end{cases}$$

or, alternatively as log-odds-ratio

$$\log \frac{\mathbb{P}\left(y = j \mid x^{(1)}, x^{(2)}\right)}{\mathbb{P}\left(y = 1 \mid x^{(1)}, x^{(2)}\right)} = \alpha_j + \beta_j^{(1)} x^{(1)} + \beta_j^{(2)} x^{(2)}$$

more generally

$$\mathbb{P}\left(y = j \mid x^{(1)}, x^{(2)}\right) = \begin{cases} \frac{\exp\left(\alpha_j + \sum_i \beta_j^{(i)} x^{(i)}\right)}{1 + \sum_{k>1} \exp\left(\alpha_k + \sum_i \beta_k^{(i)} x^{(i)}\right)} & \text{if } j > 1 \\ \frac{1}{1 + \sum_{k>1} \exp\left(\alpha_k + \sum_i \beta_k^{(i)} x^{(i)}\right)} & \text{if } j = 1 \end{cases}$$

b) write log-likelihood function.

solution: Introduce notation $d_{ij} = I(y_i = j)$
$\Longrightarrow$

$$L(\alpha, \beta) = \prod_{i=1}^{n} \prod_{j=1}^{J} \mathbb{P}\left(y_i = j \mid x_i^{(1)}, x_i^{(2)}\right)^{d_{ij}}$$
$$= \frac{\prod_{i=1}^{n} \prod_{j=1}^{J} \exp\left(\alpha_j + \sum_i \beta_j^{(i)} x^{(i)}\right)^{d_{ij}}}{\left(1 + \sum_{k>1} \exp\left(\alpha_k + \sum_i \beta_k^{(i)} x^{(i)}\right)\right)^{n}}$$

with $\alpha_1 = \beta_1^{(i)} = 0$
$\Longrightarrow$

$$\log L(\alpha, \beta) = \sum_i^n \sum_j^n d_{ij} \left(\alpha_j + \sum_i \beta_j^{(i)} x^{(i)}\right) - n \log\left(1 + \sum_{k>1} \exp\left(\alpha_k + \sum_i \beta_k^{(i)} x^{(i)}\right)\right)$$

c) write likelihood-ratio test for hypothesis of age independence of cat
$df_{full} = 3(J-1)$, $df_{red} = 2(J-1)$

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3 multinomial for aggregate data (contingency table)

for the following data:

age-group	cat1 $\left(y^{(1)}\right)$	cat2 $\left(y^{(2)}\right)$	cat3 $\left(y^{(3)}\right)$
1	$y_1^{(1)}$	$y_1^{(2)}$	$y_1^{(3)}$
2	$y_2^{(1)}$	$y_2^{(2)}$	$y_2^{(3)}$
$\vdots$			
I	$y_I^{(1)}$	$y_I^{(2)}$	$y_I^{(3)}$

a) specify multinomial reg model and interpret the coefficients:

solution: specify multinomial reg on indicator functions $x^{(i)} = I(\text{observation} \in \text{group } (i))$

$$\mathbb{P}\left(y = j \mid x^{(2)}, \ldots, x^{(I)}\right) = \begin{cases} \frac{\exp\left(\alpha_j + \sum_{i>1} \beta_j^{(i)} x^{(i)}\right)}{1 + \sum_{k>1} \exp\left(\alpha_k + \sum_{i>1} \beta_k^{(i)} x^{(i)}\right)} & \text{if } j > 1 \\ \frac{1}{1 + \sum_{k>1} \exp\left(\alpha_k + \sum_{i>1} \beta_k^{(i)} x^{(i)}\right)} & \text{if } j = 1 \end{cases}$$

or, in ANOVA-notation $\left(\beta_j^{(i)} x^{(i)} = \beta_{ij},\ \beta_{1j} = 0\right)$

$$\mathbb{P}\left(y=j \mid \text{group}\,(i)\right) = \begin{cases} \frac{\exp(\alpha_j+\beta_{ij})}{1+\sum_{k>1}\exp(\alpha_k+\beta_{ik})} & \text{if } j>1 \\ \frac{1}{1+\sum_{k>1}\exp(\alpha_k+\beta_{ik})} & \text{if } j=1 \end{cases}$$

or, alternatively as log-odds-ratio

$$\log \frac{\mathbb{P}\left(y=j \mid \text{group}\,(i)\right)}{\mathbb{P}\left(y=1 \mid \text{group}\,(i)\right)} = \alpha_j + \beta_{ij}$$
$$\alpha_1 = \beta_{1j} = \beta_{i1} = 0$$

or, as odds-ratio:

$$\log \frac{\mathbb{P}\left(y=j \mid \text{group}\,(i)\right)}{\mathbb{P}\left(y=1 \mid \text{group}\,(i)\right)} - \log \frac{\mathbb{P}\left(y=j \mid \text{group}\,(1)\right)}{\mathbb{P}\left(y=1 \mid \text{group}\,(1)\right)} = \beta_{ij}$$

hence

$$\beta_{ij} = \log \left(\frac{\mathbb{P}\left(y=j \mid \text{group}\,(i)\right)}{\mathbb{P}\left(y=1 \mid \text{group}\,(i)\right)} \Big/ \frac{\mathbb{P}\left(y=j \mid \text{group}\,(1)\right)}{\mathbb{P}\left(y=1 \mid \text{group}\,(1)\right)} \right)$$

(see mlogit in STATA [R])

b) write likelihood function

solution: see notes on multinomial regression.

c) write likelihood-ratio test for hypothesis of age-group independenc of cat

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4. multinomial output

Interpreting output from a multinomial logit model
according to the data

```
    x
y      A  B  C  D
  Y1   9  8  4  4
  Y2   4 13  9  3
  Y3   7  3 10  5
  Y4   5  1  2 13
```

consider the model

$$\mathbb{P}\left(Y=j \mid x=k\right) = \begin{cases} \frac{1}{1+\sum_{j=2}^{J}\exp(\alpha_j+\beta_{jk})} & \text{for } j=1 \\ \frac{\exp(\alpha_j+\beta_{jk})}{1+\sum_{j=2}^{J}\exp(\alpha_j+\beta_{jk})} & \text{for } 1<j\le J \end{cases} \quad \text{with } \alpha_1 = \beta_{j1} = 0$$

$$\log \frac{\mathbb{P}(Y = j \mid x = k)}{\mathbb{P}(Y = 1 \mid x = k)} = \begin{cases} \alpha_j + \beta_{jk} & \text{if } k > 1 \\ \alpha_j & \text{if } k = 1 \end{cases}$$

hence

$$\boxed{\beta_{jk} = \begin{cases} \log \frac{p_{jk}}{p_{1k}} \Big/ \log \frac{p_{j1}}{p_{11}} & \text{if } 1 < j \leq J,\ 1 < k \leq K \\ 0 & \text{if } j = 1 \end{cases}}$$

and

$$\boxed{\alpha_j = \begin{cases} \log \frac{p_{j1}}{p_{11}} & \text{if } 1 < j \leq J \\ 0 & \text{if } j = 1 \end{cases}}$$

obtain estimates

```
multinom(formula = y ~x)

Coefficients:
   (Intercept)    xB    xC   xD
Y2       -0.81  1.30  1.62 0.52
Y3       -0.25 -0.73  1.17 0.47
Y4       -0.59 -1.49 -0.11 1.77

Std. Errors:
   (Intercept)   xB   xC   xD
Y2        0.60 0.75 0.85 0.97
Y3        0.50 0.84 0.78 0.84
Y4        0.56 1.20 1.03 0.80

Residual Deviance: 243
```

means

$$\mathbb{P}(Y = \texttt{Y4} \mid x = \texttt{xD}) = \frac{\exp(-0.59 + 1.77)}{1 + \exp(-0.81 + 0.52) + \exp(-0.25 + 0.47) + \exp(-0.59 + 1.77)} = 0.520\,81$$

$$\frac{\mathbb{P}(Y = \texttt{Y4} \mid x = \texttt{xA})}{\mathbb{P}(Y = \texttt{Y1} \mid x = \texttt{xA})} = \exp(-0.59)$$

$$\frac{\mathbb{P}(Y = \texttt{Y4} \mid x = \texttt{xD})}{\mathbb{P}(Y = \texttt{Y1} \mid x = \texttt{xD})} = \underbrace{\exp(1.77)}_{5.870\,9} \frac{\mathbb{P}(Y = \texttt{Y4} \mid x = \texttt{xA})}{\mathbb{P}(Y = \texttt{Y1} \mid x = \texttt{xA})}$$

fitted values

```
     XA   XB   XC   XD
Y1 0.36 0.32 0.16 0.16
Y2 0.16 0.52 0.36 0.12
Y3 0.28 0.12 0.40 0.20
Y4 0.20 0.04 0.08 0.52\bigskip
```

check: $p_{44} = 0.52...$
$(52/20)\,/\,(16/36) = 5.85 = (p_{44}/p_{14})\,/\,(p_{41}/p_{11}) = \exp(\beta_{44})$
$p_{41}/p_{11} = 20/36 = 0.555\,56 = \exp(-0.59)$

true values:

```
     [,1] [,2] [,3] [,4]
[1,]  0.4  0.2  0.2  0.2
[2,]  0.2  0.4  0.2  0.2
[3,]  0.2  0.2  0.4  0.2
[4,]  0.2  0.2  0.2  0.4
```