last change nov1st@18.30

MLE estimation

1) mle estimation for rare events in a fixed number of bernoulli trials

model: $y_1, \ldots, y_n$ iid p_β
where $p = \mathbb{P}(Y = 1) = 1 - \mathbb{P}(Y = 0)$ find:

1. compute the likelihood function $L(p)$
2. the asymptotic variance of the mle $\widehat{p}$
3. an asymptotic 95% conf int for p
4. a graph of the likelihood function for $\sum y = 2, 1, 0$
5. an asymptotic 95% conf for p if $n = 16$ and $\sum y = 2, 1, 0$ what do you note?
6. generalize this result

solution to 7.
solve

$$p \pm 2\sqrt{\frac{p(1-p)}{n}} < 0$$

solution:

$$p < \frac{4}{4+n}$$

example: $n = 16$. for $p < 0.20$ we expect a 95% asymptotic confidence interval to include 0 in its interior!

2) Reparametrized Model:

model: $y_1, \ldots, y_n$ iid p_β
where $p_\beta = \mathbb{P}(Y = 1) = 1 - \mathbb{P}(Y = 0) = \frac{\exp\beta}{1+\exp\beta}$ find:

1. compute the log likelihood function for β
2. $\widehat{\beta}$ mle of β.
3. the Fisher information

4. the asymptotic distribution of $\widehat{\beta}$
5. estimate the asymptotic variance of $\widehat{\beta}$
6. estimate an asymptotic 95% confidence interval for β
7. from 6. derive an asymptotic 95% confidence interval for p

sol: see this transformation $f(\beta) = \frac{\exp\beta}{1+\exp\beta}$on *www.wolframalpha.com*

$$
\begin{aligned}
L(\beta) &= \frac{\exp\beta^{\sum y}}{(1+\exp\beta)^n} \\
l(\beta) &= \beta\sum y - n\log(1+\exp\beta) \\
\frac{dl}{d\beta} &= \sum y - \frac{n\exp\beta}{1+\exp\beta} = 0 \\
\frac{d^2 l}{d\beta^2} &= -n\frac{\exp\beta(1+\exp\beta) - 2\exp\beta}{(1+\exp\beta)^2} = -n\frac{\exp\beta}{(1+\exp\beta)^2}
\end{aligned}
$$

$\Longrightarrow$ **MLE** if[1] $i = 1, \ldots, n$

$$\widehat{\beta} = \log\frac{\sum y}{n - \sum y}$$

asymptotic variance:

$$\mathbb{V}\left(\widehat{\beta}\right) = \frac{(1+\exp\beta)^2}{n\exp\beta}$$

asympt CI: $\widehat{\beta} \pm 2\sqrt{\mathbb{V}\left(\widehat{\beta}\right)}$

$$\widehat{\beta} \pm 4\sqrt{\frac{\cosh\widehat{\beta}}{n}}$$

because $\sqrt{n\mathbb{V}\left(\widehat{\beta}\right)} = \sqrt{\frac{(1+\exp\beta)^2}{\exp\beta}} = \frac{(1+\exp\beta)}{\exp\beta/2} = 2\frac{\exp(-\beta/2)+\exp(\beta/2)}{2} = 2\cosh(\beta)$

hence, back transforming, a 95% asymptotic conf int for p :

$$\frac{\exp\widehat{\beta}_{\text{lo}}}{1+\exp\widehat{\beta}_{\text{lo}}} < p < \frac{\exp\widehat{\beta}_{\text{hi}}}{1+\exp\widehat{\beta}_{\text{hi}}}$$

[1] It is very easy to see:
$L(y_0) = \max_y L(y)$ and $y = g(x) \Longrightarrow L(g(x_0)) = \max_x L(g(x))$ with $y_0 = g(x_0)$

and, naturally

$$0 < \frac{\exp \widehat{\beta}_{\mathrm{lo}}}{1 + \exp \widehat{\beta}_{\mathrm{lo}}} < \frac{\exp \widehat{\beta}_{\mathrm{hi}}}{1 + \exp \widehat{\beta}_{\mathrm{hi}}} < 1$$

3) generalization:

model: $y_1, \ldots, y_n, y_{n+1}, \ldots, y_{n+m}$ iid p
where

$$p_\beta = \mathbb{P}\left(Y_i = 1\right) = 1 - \mathbb{P}\left(Y_i = 0\right) = \begin{cases} \frac{\exp \beta}{1+\exp \beta} & \text{if } i = 1, \ldots, n \\ \frac{\exp(\beta+\delta)}{1+\exp(\beta+\delta)} & \text{if } i = n+1, \ldots, n+m \end{cases}$$

find:

1. the log likelihood function for (β, δ)
2. $\widehat{\beta}$ mle of β, $\widehat{\delta}$ mle of δ.
3. the Fisher information
4. the asymptotic distribution of $\left(\widehat{\beta}, \widehat{\delta}\right)$
5. estimate the asymptotic covariance matrix of $\left(\widehat{\beta}, \widehat{\delta}\right)$
6. estimate an asymptotic 90% confidence interval for β
7. test for $\mathrm{H}_0 : \delta = 0$